A Trig Identities

A.1. All of the trigonometric functions of an angle θ can be constructed geometrically in terms of a unit circle centered at origin as shown in Figure 87.

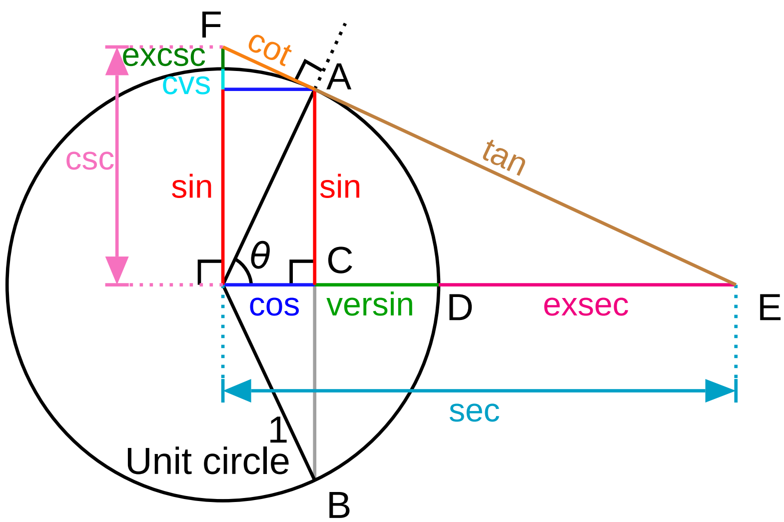

Figure 87: Trigonometric functions on a unit circle.

A.2. Cosine function

(a) Is an even function: $\cos(-x) = \cos(x)$.

(b) $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$.

(c) Sum formula:

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y. \tag{98}$$

(d) Product-to-Sum Formula:

$$\cos(x)\cos(y) = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right).$$

(e) $$\cos^n x = \begin{cases} \frac{1}{2^{n-1}} \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{k} \cos\left((n-2k)\,x\right), & \text{odd } n \geq 1 \\ \frac{1}{2^n} \left(\sum_{k=0}^{\frac{n}{2}-1} 2\binom{n}{k} \cos\left((n-2k)\,x\right) + \binom{n}{\frac{n}{2}} \right), & \text{even } n \geq 2 \end{cases}$$

(f) Any two real numbers a, b can be expressed in terms of cosine and sine with the same amplitude and phase:

$$(a, b) = (A\cos(\phi), A\sin(\phi)), \tag{99}$$

where $A = \sqrt{a^2 + b^2}$ and $\phi = \text{atan2}(b, a)$. This is simply the polar-coordinates from of the point (a, b) on Cartesian coordinates. Note that atan2 is the four quadrant inverse tangent which takes the signs of a and b into account to determine in which quadrant ϕ lies.

A.3. The complex exponential function $e^{j\theta}$

(a) As a function of θ, $e^{j\theta}$ is periodic with period 2π.

(b) ***Euler's formula***: $\boxed{e^{j\theta} = \cos\theta + j\sin\theta.}$

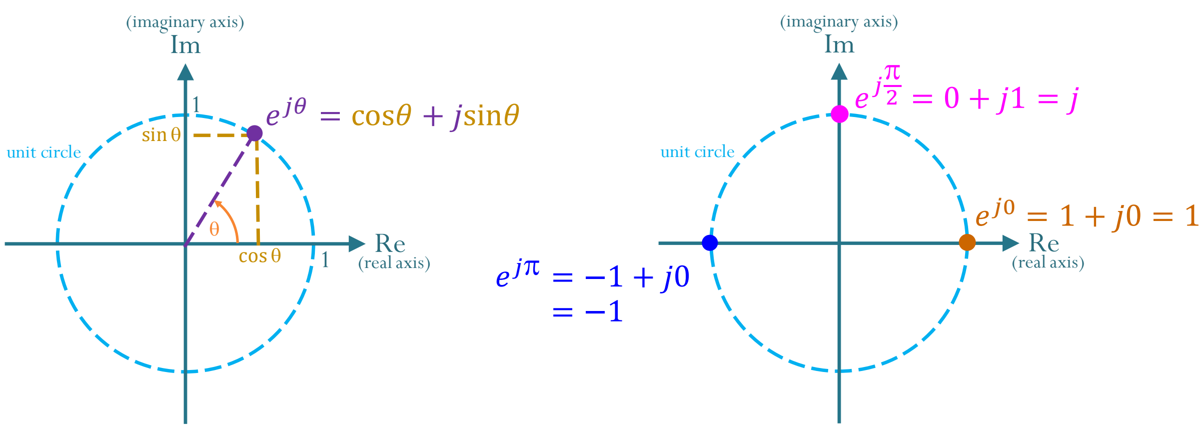

Figure 88: Euler's Formula on the Complex Plane

$$\cos(\theta) = \text{Re}\left\{e^{j\theta}\right\} = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$
$$\sin(\theta) = \text{Im}\left\{e^{j\theta}\right\} = \text{Re}\left\{-je^{j\theta}\right\} = \text{Re}\left\{+\frac{1}{j}e^{j\theta}\right\} = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right).$$

(c) Any complex number $z = x + jy$ can be expressed as

$$z = \sqrt{x^2 + y^2}e^{j\phi} = |z|e^{j\phi},$$

where $\phi = \text{atan2}(b, a)$.

- $z^t = |z|^t e^{j\phi t}$.

(d) Using the Euler's formula, we can rewrite a linear combination of two cosines as a single cosine:

$$\begin{aligned} a_1 \cos(\theta_1) + a_2 \cos(\theta_2) &= a_1 \operatorname{Re}\left\{e^{j\theta_1}\right\} + a_2 \operatorname{Re}\left\{e^{j\theta_2}\right\} \\ &= \operatorname{Re}\left\{a_1 e^{j\theta_1} + a_2 e^{j\theta_2}\right\}. \end{aligned}$$

The sum $a_1 e^{j\theta_1} + a_2 e^{j\theta_2}$ is a sum of two complex numbers and hence is another complex number. Suppose the polar form of this number is $ae^{j\theta}$. Then,

$$a_1 \cos(\theta_1) + a_2 \cos(\theta_2) = \operatorname{Re}\left\{ae^{j\theta}\right\} = a\cos\theta.$$

(e) Using the Euler's formula, we can rewrite linear combination of cosine and sine of the same argument as a single cosine by

$$a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\cos(\theta-\phi),$$

where $\phi = \operatorname{atan2}(b,a)$. To see this, note that

$$a\cos\theta + b\sin\theta = \operatorname{Re}\left\{ae^{j\theta}\right\} + \operatorname{Re}\left\{-jbe^{j\theta}\right\} = \operatorname{Re}\left\{(a-jb)\,e^{j\theta}\right\}.$$

We know that $a+jb = \sqrt{a^2+b^2}e^{j\phi}$ where $\phi = \operatorname{atan2}(b,a)$. So, $a-jb = (a+jb)^* = \sqrt{a^2+b^2}e^{-j\phi}$. Hence,

$$a\cos\theta + b\sin\theta = \operatorname{Re}\left\{\sqrt{a^2+b^2}e^{-j\phi}e^{j\theta}\right\}$$

Another way to see this is to re-express the two real numbers a, b using (99) and then use (98).

(f) More relations involving sin and cos.

- $e^{jAt} + e^{jBt} = 2e^{j\frac{A+B}{2}t}\cos\left(\frac{A-B}{2}\right)$.
- $e^{jAt} - e^{jBt} = 2je^{j\frac{A+B}{2}t}\sin\left(\frac{A-B}{2}\right)$
- $\frac{e^{jAt}-e^{jBt}}{e^{jCt}-e^{jDt}} = e^{j\frac{(A+B)-(C+D)}{2}t}\frac{\sin\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C-D}{2}\right)}$.